ϕ_p = volume of particles per unit volume of fluid-particle mixture

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A Unified Treatment of Drainage, Withdrawal, and Postwithdrawal Drainage with Inertial Effects

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The aim of this note is twofold: first, we attempt a unified description of the drainage, withdrawal, and postwithdrawal drainage of a liquid over a flat plate; second, we examine the question of which of two conditions that have appeared in the literature is to be employed for correctly obtaining the film thickness profiles. A knowledge of these profiles is of importance in a variety of applications, such as dipcoating, enameling, electroplating, and capillary viscometry (Tallmadge and Gutfinger, 1967).

PLATE LIFTED WITH A GENERAL VELOCITY

We choose the yz and xz planes to coincide with the initial position of the plate and the initial surface of the bath. At $t = 0$, the plate is lifted along the y axis with a moderate velocity $f(t)$, and the bath is allowed to drain freely under the action of gravity. The solution of the problem will be obtained for arbitrary $f(t)$. From this, the results for the drainage, withdrawal, and post withdrawal processes emerge as special cases for appropriate choices of the plate velocity $f(t)$.

We confine ourselves to the parallel flow region and neglect the effect of surface tension on the flow. With the assumption of laminar one-dimensional flow, permissible in this region (Groenveld, 1970), the Navier-Stokes equation in the y direction reduces to (Gutfinger and Tallmadge, 1964)

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} - g \quad (1)$$

The velocity $v(x, t)$ of the liquid is subject to the usual conditions

$$v(x, 0) = 0, \quad \frac{\partial v}{\partial x}(h, t) = 0 \quad \text{and} \quad v(0, t) = f(t) \quad (2)$$

Taking the Laplace transform of Equation (1), we obtain

$$\frac{d^2\bar{v}}{dx^2} - \frac{s}{\nu}\bar{v} = \frac{g}{s\nu} \quad (3)$$

where $\bar{v}(x, s)$ is required to satisfy the conditions

$$\bar{v}(0, s) = \bar{f}(s) \quad \text{and} \quad \frac{d\bar{v}}{dx} = 0 \quad \text{at} \quad x = h \quad (4)$$

Equation (3) has the solution

$$\bar{v}(x, s) = \left[\bar{f}(s) + \frac{g}{s^2}\right] \frac{\cosh\sqrt{\frac{s}{\nu}}\,(x-h)}{\cosh\sqrt{\frac{s}{\nu}}\,h} - \frac{g}{s^2} \quad (5)$$

The inverse of Equation (5) gives the solution of Equation (1):

$$v(x, t) = \frac{g}{2\nu}(x^2 - 2xh) + \frac{2gh^2}{\nu} \sum_{n=0}^{\infty} \frac{e^{-\alpha_n^2 \nu t/h^2}}{\alpha_n^3} \sin\frac{\alpha_n x}{h} + \frac{2\nu}{h^2} \sum_{n=0}^{\infty} \alpha_n e^{-\alpha_n^2 \nu t/h^2} \sin\frac{\alpha_n x}{h} \cdot \int_0^t f(w) e^{\alpha_n^2 \nu w/h^2} dw,$$

$$x > 0 \quad = f(t), \quad \text{for} \quad x = 0 \quad (6)$$

where $\alpha_n = (n + 1/2)\pi$, and w is the supplementary variable brought in by the convolution theorem. In terms of nondimensional quantities, the flux of the entrained liquid is obtained as

$$Q = \int_0^H V(X, T)\,dX = -\frac{H^3}{3} + 2H^3 \sum_{n=0}^{\infty} \frac{e^{-\alpha_n^2 T/H^2}}{\alpha_n^4} + \frac{2}{H} \sum_{n=0}^{\infty} e^{-\alpha_n^2 T/H^2} \cdot \left[\int_0^T F(W) e^{\alpha_n^2 W/H^2} dW\right] \quad (7)$$

The film thickness profile is connected to the flow rate through the equation of continuity, which yields

$$Y = \int \left(\frac{\partial Q}{\partial H}\right)_T dT + \phi(H) \quad (8)$$

In order to evaluate the integration 'constant' $\phi(H)$, we have to impose an additional condition on Equation (8). Two such conditions occur in the literature. Gutinger and Tallmadge (1964) employed the boundary condition $H(T) = 0$ at the top of the film, $Y = Y_o$. On the other hand, an initial condition, $H = \infty$ at $T = 0$, was implicit in the original work of Jeffreys (1930) (Van Rossum, 1958) and was explicitly stated by Lang and Tallmadge (1971). It is important to decide which condition to use, as the resulting film profiles are different from one another. The choice is facilitated by the following considerations:

1. Involving, as it does, an integration over time, Equation (8) calls for an initial condition rather than a boundary condition.
2. The boundary condition $H(T) = 0$ at $Y = Y_o$ fails to hold at $T = 0$ (when the surface of the bath is strictly horizontal) and thus contradicts the initial condition $H = \infty$ at $T = 0$.
3. On the other hand, the initial condition $H = \infty$ at $T = 0$ results in a film profile which satisfies the boundary condition $H(T) = 0$ at $Y = Y_o$ for $T > 0$ [see Equation (12)].
4. Only the profile obtained from the initial condition correctly describes the descent of the liquid far away from the plate [see Equation (11)].

In view of these considerations, we prefer to use the initial condition as the more justified in the context and obtain the implicit expression

$$Y = -H^2T + \frac{2}{3}H^4 - 2H^2 \sum_{n=0}^{\infty} \frac{e^{-\alpha_n^2 T/H^2}}{\alpha_n^6}(5H^2 + 2\alpha_n^2 T) + \int_0^T F(T)dT - \frac{2}{H^2}\sum_{n=0}^{\infty}\frac{e^{-\alpha_n^2T/H^2}}{\alpha_n^2}(2\alpha_n^2 T + H^2)\int_0^T F(W)e^{\alpha_n^2 W/H^2}dW + \frac{4}{H^2}\sum_{n=0}^{\infty} e^{-\alpha_n^2 T/H^2}\left[\int_0^T WF(W)e^{\alpha_n^2 W/H^2}dW\right] \quad (9)$$

giving the film thickness profiles for a general plate velocity $f(t)$.

DRAINAGE

With $f(t) = 0$, the foregoing equations describe the draining of a flat plate. Equation (9) then reduces to

$$Y = -H^2T + \frac{2}{3}H^4 - 2H^2 \sum_{n=0}^{\infty} \frac{e^{-\alpha_n^2 T/H^2}}{\alpha_n^6}(5H^2 + 2\alpha_n^2 T) \quad (10)$$

If inertial terms were ignored in Equation (1), one would obtain the Jeffreys' parabola $Y = -H^2T$. The second and third terms of Equation (10) therefore represent the effect of the inertial terms on the film thickness profile. A curious feature of Equation (10) is that it does not reduce to Jeffreys' expression as $T \to \infty$ owing to the presence of the second term. However, this hardly constitutes a tenable argument against it. For we can only demand that the solution (6) of the Navier-Stokes Equation (1) should reduce to its steady state value as $T \to \infty$. This it does, as can easily be verified. On the other hand, Equation (10) for the film thickness profile involves a further integration (8) over time of the solution (6). This would bring in additional terms representing the accumulated effect of the transients in (6). The constant 2/3 H^4 of Equation (10) owes its origin to this integration and amounts to an inertial correction to the Jeffreys' parabola in the steady state. It is not of much importance in the parallel flow region, which is characterized by small values of H.

On the other hand, its presence is indispensable if Equation (10) is to behave properly as $H \to \infty$. In this limit, we have

$$Y = -H^2T + \frac{2}{3}H^4 - 2H^2\sum_{n=0}^{\infty}\frac{1}{\alpha_n^6}\left[1 - \frac{\alpha_n^2T}{H^2} + \frac{\alpha_n^4T^2}{2H^4} - ..\right](5H^2 + 2\alpha_n^2T) = -\frac{T^2}{2} \quad (11) \quad \text{(Jolley, 1960)}.$$

(11)

This is equivalent to $y = -1/2\ gt^2$, which correctly describes the free fall of the bath far away from the plate. In the absence of the second term 2/3 H^4, the expression for Y would diverge and tend to $-\infty$ as $H \to \infty$. Rewriting Equation (10) as

$$Y = H^2\Psi(H, T) \quad (12)$$

where

$$\Psi(H,T) = -T + \frac{2}{3}H^2 - 2\sum_{n=0}^{\infty}\frac{e^{-\alpha_n^2T/H^2}}{\alpha_n^6}(5H^2 + 2\alpha_n^2T)$$

we note that for $T > 0$, $Y = 0$ implies $H = 0$. [It can be verified from a graphical solution that $\Psi(H, T)$ has no root for real values of H.] This ensures that the film profile described by Equation (10) not only reduces to the flat surface of the bath at $T = 0$ but also satisfies the condition $H = 0$ at $Y = 0$ of Gutfinger and Tallmadge (1964) for $T > 0$. Although it meets all these requirements of the one-dimensional model, Equation (10) neglects the effects of surface tension and is therefore not expected to accurately depict the dynamic meniscus region.

WITHDRAWAL

The unsteady film profile in continuous lifting can be obtained by taking $f(t) = v_o$ in Equation (9). We then obtain

$$Y = -H^2T + \frac{2}{3}H^4 - V_oH^2 + V_oT - 2\sum_{n=0}^{\infty}\frac{e^{-\alpha_n^2T/H^2}}{\alpha_n^6}[H^2(5H^2 + 2\alpha_n^2T) - \alpha_n^2V_o(3H^2 + 2\alpha_n^2T)] \quad (13)$$

By dividing both sides by T and by taking the limit as $T \to \infty$, Equation (13) is reduced to the well-known expression for the constant film thickness at high capillary numbers [Groenveld, 1970; Spiers et al., 1974]

$$H_o = \sqrt{V_o} \quad (14)$$

with a corresponding steady state flux

$$Q_o = \frac{2}{3}V_o^{3/2} \quad (15)$$

It is interesting to note that the inertial terms in the film profile (13) of continuous withdrawal drop away in the steady state.

POSTWITHDRAWAL DRAINAGE

This process can be described by the step function $f(t) = v_o$ for $0 < t < t_w$ and $f(t) = 0$ for $t > t_w$. With this choice, Equation (9) yields, as $T \rightarrow \infty$

$$Y = -H^2T + \frac{2}{3}H^4 + V_oT_w \qquad (16)$$

Apart from the inertial correction $2/3\ H^4$, which again makes its appearance, it can easily be checked that this is precisely the steady state postwithdrawal profile of Lang and Tallmadge (1971).

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NOTATION

Lengths and times (denoted by lower case letters) are to be multiplied, respectively, by the factors $(g/\nu^2)^{1/3}$ and $(g^2/\nu)^{1/3}$ in order to obtain the corresponding dimensionsionless quantities (denoted by capitals).

g = acceleration due to gravity
h = liquid film thickness
Q = flux of the fluid entrained
t_w = withdrawal time in post withdrawal drainage
v = y component of the fluid velocity
v_o = constant withdrawal speed
Y_o = y coordinate of the top of the film
ν = kinematic viscosity of the fluid

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The Effect of Cross Sectional Pore Shape on Knudsen Diffusion in Porous Materials

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The ideal model of a porous material would somehow combine the contradictory attributes of physical realism and mathematical tractability. Of the many attempts to provide such a model, the most popular has been the visualization of a porous material as a solid pierced by nonintersecting, randomly oriented, cylindrical pores possessing diameters ranging over perhaps several orders of magnitude. This model has achieved a reasonable success both in pore structure studies (Dullien and Batra, 1970) and in transport studies (Satterfield, 1970). Yet it is recognized that this model can have severe limitations, and other models are being proposed continually to remedy the defects in the cylindrical pore model. The Dullien-Batra review gives an account of many of those offered in pore structure studies, and examples of other models for transport are presented by Flood et al. (1952), Dullien (1975), and Neal and Nader (1976).

But before a defect is considered sufficiently serious to require a remedy, the probable error it will cause should be estimated if possible. For example, the presence of pore constrictions has been shown to be a serious factor in predicting transport rates through porous materials from both theoretical considerations (Petersen, 1958; Michaels, 1959; Haynes and Brown, 1971) and experimental implications (Brown et al., 1969). On the other hand, pore constrictions are not indicated as a serious problem when gaseous diffusion rates are predicted at one pressure from those measured at another pressure (Haynes and Brown, 1971).

Another possible flaw in the cylindrical pore model arises from the presence of pores with noncircular cross sections in real materials. While pores with circular cross sections are occasionally observed (Adair et al., 1972), the majority of porous materials appears to possess pores with highly irregular shapes (for example, Dullien and Mehta, 1972; Eklund, 1976). This note considers the effect of noncircular cross-sectional pores on transport through porous materials.

Most transport situations within porous materials involve one or more of three types of flow: bulk flow in the laminar flow regime, molecular diffusive flow, and Knudsen flow. In solids whose pores are not too small, combinations of the appropriate transport laws appear to predict the behavior very well (Mason et al., 1967; Gunn and King, 1969). The lower limit of pore size to which these laws can be applied appears to be about 50Å in radius (Omata and Brown, 1972), although it may be

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